Homogeneous Levi degenerate CR-manifolds in dimension 5

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1. Introduction

Levi nondegenerate real-analytic hypersurfaces $M \subset \mathbb{C}^n$ are well understood due to the seminal results of Tanaka [17] and Chern-Moser[5], where a complete set of *local* invariants for M has been defined. At the other extreme are those hypersurfaces which are locally CR-equivalent to a product $M' \times \mathbb{C}$, (take the real hyperplanes in $\mathbb{C}^n$ as a simple example). In-between there is a huge and much less understood class of CR-hypersurfaces which are neither Levi nondegenerate nor locally direct products as above. A convenient description of that class of CR-manifolds can be given in terms of k-nondegeneracy in the sense of [3] with $k \geq 2$, see [2] for further details. The (uniformly) 2-nondegenerate real-analytic hypersurfaces are the main focus of this paper.

Clearly, 5 is the lowest dimension of M (that is, $n = 3$) for which Levi degenerate hypersurfaces can occur that are not locally equivalent to $M' \times \mathbb{C}$ for any CR-manifold M'. Of a particular interest are those manifolds that are Levi degenerate at every point. A well studied example of this type is the tube over the future light cone in 3-dimensional space-time, more precisely

$$\mathcal{M} := \{z \in \mathbb{C}^3 : (\mathrm{Re}\, z_1)^2 + (\mathrm{Re}\, z_2)^2 = (\mathrm{Re}\, z_3)^2,\ \mathrm{Re}(z_3) > 0\}\,,$$

compare e.g. [8], [10], [14] and [16]. $\mathcal{M}$ is nondegenerate in a higher order sense (2-nondegenerate, to be precise) and is also homogeneous (a 7-dimensional group of complex affine transformations acts transitively on it). All further examples of locally homogeneous 2-nondegenerate CR-manifolds of dimension 5 known so far finally turned out to be locally CR-isomorphic to the light cone tube $\mathcal{M}$, compare e.g. [9], [14], [12], [10], and the belief arose that there are no others. The main objective of this paper is the presentation of an infinite family of homogeneous 2-nondegenerate CR-manifolds of dimension 5 which are pairwise locally CR-inequivalent. In the forthcoming paper [11] it will be shown by purely Lie algebraic methods that our list of examples is complete in the following sense: *Every locally homogeneous 2-nondegenerate real-analytic CR-manifold of dimension 5 is locally CR-equivalent to one of our examples.*

A main point in our discussion is to show that the provided examples (Examples 6.1 – 6.6) are mutually CR-nonequivalent. It should be noted that even in case of Levi nondegenerate hypersurfaces it is in general quite hard to decide whether these are locally CR-equivalent at selected points, although this is possible in principle due to the existence of a complete system of invariants [5]. In the Levi degenerate case much less is known and suitable invariants have to be created ad hoc. In our particular case of locally homogeneous CR-manifolds M we use for every $a \in M$ as invariant the Lie algebra isomorphy type of $\mathfrak{hol}(M, a)$, which by definition is the real Lie algebra of all germs at a of local infinitesimal CR-transformations defined in suitable neighbourhoods of $a \in M$, see Section 2 for details. It is meanwhile well known that for the tube $\mathcal{M}$ over the future light cone in $\mathbb{R}^3$ the Lie algebras $\mathfrak{hol}(\mathcal{M}, a)$ have dimension 10 and are isomorphic to the simple Lie algebra $\mathfrak{so}(2,3)$. It turns out that for all our examples

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$M \neq \mathcal{M}$ of 2-nondegenerate locally homogeneous CR-manifolds the Lie algebras $\mathfrak{hol}(M,a)$ are solvable of dimension 5 and are mutually non-isomorphic for different M. Clearly, this procedure needs a feasible way to compute the Lie algebras $\mathfrak{hol}(M,a)$ explicitly. One of our major goals is to provide such a way at least for manifolds of tube type.

The paper is organized as follows. After recalling some necessary preliminaries in Section 2 we discuss in Section 3 tube manifolds $M = F \oplus i\mathbb{R}^n \subset \mathbb{C}^n$ over real-analytic submanifolds $F \subset \mathbb{R}^n$. It turns out that the CR-structure of M is closely related to the real-affine structure of the base F. For instance, the Levi form of M is essentially the sesquilinear extension of the second fundamental form of the submanifold $F \subset \mathbb{R}^n$. Generalizing the notion of the second fundamental form we define higher order invariants for F (see 3.3). In the uniform case these invariants precisely detect the k-nondegeneracy of the corresponding CR-manifold $M = F \oplus i\mathbb{R}^n$. It is known that the (uniform) k-nondegeneracy of a real-analytic CR-manifold M together with minimality ensures that the Lie algebras $\mathfrak{hol}(M,a)$ are finite-dimensional. For submanifolds $F \subset \mathbb{R}^n$, homogeneous under a group of affine transformations, a simple criterion for the 2-nondegeneracy of the associated tube manifold M is given in Proposition 3.6. We close this section with some general remarks concerning polynomial vector fields.

In Section 4 these results are applied to the case where F is conical in $\mathbb{R}^n$, that is, locally invariant under dilations $z \mapsto tz$ for t near $1 \in \mathbb{R}$. In this case M is always *Levi degenerate*. Assuming that $\mathfrak{hol}(M,a)$ is finite dimensional (equivalently, M is holomorphically non-degenerate and minimal) we develop some basic techniques which enable us to compute $\mathfrak{hol}(M,a)$. Our first observation is that under the finiteness assumption $\mathfrak{hol}(M,a)$ consists only of polynomial vector fields and carries a natural graded structure, see Lemma 4.1. We prove further (under the same assumptions, see Proposition 4.3) that local CR-equivalences between two such tube manifolds are always rational (even if these manifolds are not real-algebraic). If all homogeneous parts in $\mathfrak{hol}(M,a)$ of degree ≥ 2 vanish, this result can be further strengthened, see Proposition 4.6. We close the section with an investigation of the tubes $M^\alpha_{p,q}$ over the cones $F^\alpha_{p,q} := \{x \in \mathbb{R}^{p+q}_+ : \sum \varepsilon_j x_j^\alpha = 0\}$ with p positive and q negative ε_j's and $\alpha \in \mathbb{R}^*$. We show how the preceeding results can be used to determine explicitly all $\mathfrak{hol}(M^\alpha_{p,q}, a)$ for $\alpha \geq 2$ an arbitrary integer.

In Section 5, we study homogeneous CR-submanifolds $M^\varphi \subset \mathbb{C}^n$ depending on the choice of an endomorphism $\varphi \in \mathrm{End}(\mathbb{R}^n)$ together with a cyclic vector $a \in \mathbb{R}^n$ for φ in the following way: The powers $\varphi^0, \varphi^1, \ldots, \varphi^d$ $(2 \leq d+1 < n)$ span an abelian Lie algebra $\mathfrak{h}^\varphi$ and, in turn, give the cone $F^\varphi = \exp(\mathfrak{h}^\varphi)a \subset \mathbb{R}^n$. For 'generic' φ the corresponding tube manifolds $M^\varphi = F^\varphi \oplus i\mathbb{R}^n$ are 2-nondegenerate and of CR-dimension $d+1$. In this case the local invariants $\mathfrak{hol}(M^\varphi, a)$ are explicitly determined (Proposition 5.3). Further, again for φ in general position the tube manifold M^φ is simply connected and has trivial stability group at every point. As a consequence, the manifolds $M = M^\varphi$ of this type have the following remarkable property: *Every homogeneous (real-analytic) CR-manifold locally CR-equivalent to M is globally CR-equivalent to M* (see 5.8). Specialized to $n = 3$ we get in Section 6 an infinite family of homogeneous 2-nondegenerate hypersurfaces in $\mathbb{C}^3$. These are all tubes over linearly homogeneous cones in $\mathbb{R}^3$. Besides these, there is just one other example, namely the tube over the twisted cubic in $\mathbb{R}^3$ (which is not a cone but is affinely homogeneous).

2. Preliminaries

In the following let E always be a complex vector space of finite dimension and $M \subset E$ a locally closed connected real-analytic submanifold (unless stated otherwise). Due to the canonical identifications $T_aE = E$, for every $a \in M$ we consider the tangent space T_aM as an $\mathbb{R}$-linear subspace of E. Set $H_aM := T_aM \cap iT_aM$. The manifold M is called a *CR-submanifold* if the dimension of H_aM does not depend on $a \in M$. This complex dimension is called the *CR-dimension* of M and H_aM is called the *holomorphic tangent space* at a, compare [2] as general reference for CR-manifolds. Given a further real-analytic CR-submanifold M' of a complex vector space E' a smooth mapping $h : M \to M'$ is called *CR* if for all $a \in M$ the differentials $dh_a : T_aM \to T_{ha}M'$ map the corresponding holomorphic

tangent spaces in a complex linear way to each other. A vector field on M is a mapping $f : M \to E$ with $f(a) \in T_aM$ for all $a \in M$. For better distinction we also write $\xi = f(z)\partial/_{\partial z}$ instead of f and ξ_a instead of $f(a)$, compare the convention (2.1) in [10]. The real-analytic vector field $\xi = f(z)\partial/_{\partial z}$ is called an *infinitesimal CR-transformation* on M if the corresponding local flow consists of CR-transformations.

For simplicity and without essential loss of generality we always assume that the CR-submanifold M is *generic* in E, that is, $E = T_aM + iT_aM$ for all $a \in M$. Given an infinitesimal CR-transformation $f(z)\partial/_{\partial z}$ on a generic CR-submanifold M the map f extends uniquely to a holomorphic map $f : U_f \to E$ in an open neighborhood U_f of M in E, (see [1] or 12.4.22 in [2]). Let us denote by $\mathfrak{hol}(M)$ the space of all such vector fields, which is a real Lie algebra with respect to the usual bracket. Further, for every $a \in M$ we denote by $\mathfrak{hol}(M, a)$ the Lie algebra of all germs of infinitesimal CR-transformations defined in arbitrary open neighbourhoods of $a \in M$. Since M is generic in E the space $\mathfrak{hol}(M, a)$ naturally embeds as a real Lie subalgebra of the complex Lie algebra $\mathfrak{hol}(E, a)$. The CR-manifold M is called *holomorphically nondegenerate* at a if $\mathfrak{hol}(M, a)$ is totally real in $\mathfrak{hol}(E, a)$, that is, if $\mathfrak{hol}(M, a) \cap i\,\mathfrak{hol}(M, a) = 0$ in $\mathfrak{hol}(E, a)$. This condition together with the minimality of M in E are equivalent to $\dim \mathfrak{hol}(M, a) < \infty$ (see (12.5.16) in [2]). Here, the CR-submanifold $M \subset E$ is called *minimal* at $a \in M$ if $T_aR = T_aM$ for every submanifold $R \subset M$ with $a \in R$ and $H_aM \subset T_aR$.

By $\mathfrak{aut}(M, a) := \{\xi \in \mathfrak{hol}(M, a) : \xi_a = 0\}$ we denote the *isotropy subalgebra* at $a \in M$. Clearly, $\mathfrak{aut}(M, a)$ has finite codimension in $\mathfrak{hol}(M, a)$. Furthermore, we denote by $\mathrm{Aut}(M, a)$ the group of all *germs* of real-analytic CR-isomorphisms $h : W \to \widetilde{W}$ with $h(a) = a$, where $W, \widetilde{W}$ are arbitrary open neighbourhoods of a in M. It is known that every germ in $\mathrm{Aut}(M, a)$ can be represented by a holomorphic map $U \to E$, where U is an open neighbourhood of a in E, compare e.g. 1.7.13 in [2]. Finally, $\mathrm{Aut}(M)$ denotes the group of all global real-analytic CR-automorphisms $h : M \to M$ and $\mathrm{Aut}(M)_a$ its isotropy subgroup at a. There is a canonical group monomorphism $\mathrm{Aut}(M)_a \hookrightarrow \mathrm{Aut}(M, a)$ as well as an exponential map $\exp : \mathfrak{aut}(M, a) \to \mathrm{Aut}(M, a)$ for every $a \in M$.

A basic invariant of a CR-manifold is the (vector valued) *Levi form*. Its definitions found in the literature may differ by a constant factor. Here we choose the following definition: It is well-known that for every point a in the CR-manifold M there is a well defined alternating $\mathbb{R}$-bilinear map

$$\omega_a : H_aM \times H_aM \;\to\; E/H_aM$$

satisfying $\omega_a(\xi_a, \zeta_a) = [\xi, \zeta]_a \mod H_aM$, where ξ, ζ are arbitrary smooth vector fields on M with $\xi_z, \zeta_z \in H_zM$ for all $z \in M$. We define the Levi form

$$\mathcal{L}_a : H_aM \times H_aM \;\to\; E/H_aM \tag{2.1}$$

to be twice the sesquilinear part of ω_a. By *sesquilinear* we always mean 'conjugate linear in the first and complex linear in the second variable', that is,

$$\mathcal{L}_a(v, w) = \omega_a(v, w) + i\omega_a(iv, w) \qquad \text{for all } v, w \in H_aM\,.$$

In particular, the vectors $\mathcal{L}_a(v, v)$, $v \in H_aM$, are contained in iT_aM/H_aM, which can be identified in a canonical way with the normal space E/T_aM to $M \subset E$ at a.

Define the *Levi kernel*

$$K_aM := \{v \in H_aM : \mathcal{L}_a(v, w) = 0 \;\text{ for all }\; w \in H_aM\}\,.$$

The CR-manifold M is called *Levi nondegenerate* at a if $K_aM = 0$. Generalizing that, the notion of k-*nondegeneracy* for M at a has been introduced for every integer $k \geq 1$ (see [3], [2]). As shown in 11.5.1 of [2] a real-analytic and connected CR-manifold M is holomorphically nondegenerate at a (equivalently:

at every $z \in M$) if and only if M is k-nondegenerate at a for some $k \geq 1$. For $k = 1$ this notion is equivalent to M being Levi nondegenerate at $a \in M$.

In Section 5 we also need a more general notion of (real-analytic) CR-manifold. This is a connected real-analytic manifold M together with a subbundle $HM \subset TM$ and a bundle endomorphism $J : HM \to HM$ satisfying the following property: For every point of M a suitable open neighbourhood can be realized as (locally closed) CR-submanifold U of some $\mathbb{C}^n$ in such a way that H_zM corresponds to $H_zU = T_zU \cap iT_zU$ and the restriction of J to H_zM corresponds to the multiplication with the imaginary unit i on H_zU for every $z \in U \subset M$.

3. Tube manifolds

Let V be a real vector space of finite dimension and $E := V \oplus iV$ its complexification. Let furthermore $F \subset V$ be a connected real-analytic submanifold and $M := F \oplus iV \subset E$ the corresponding tube manifold. M is a generic CR-submanifold of E invariant under all translations $z \mapsto z + iv$, $v \in V$. In case V' is another real vector space of finite dimension, E' is its complexification, $F' \subset V'$ a real-analytic submanifold and $\varphi : V \to V'$ an affine mapping with $\varphi(F) \subset F'$, then clearly φ extends in a unique way to a complex affine mapping $E \to E'$ with $\varphi(M) \subset M'$. It should be noted that higher order real-analytic maps $\psi : F \to F'$ also extend locally to holomorphic maps $\psi : U \to E'$, U open in E. But in contrast to the affine case we have in general $\psi(M \cap U) \not\subset M'$. We may therefore ask how the CR-structure of M is related to the real affine structure of the submanifold $F \subset V$.

For every $a \in F$ let $T_aF \subset V$ be the *tangent space* and $N_aF := V/T_aF$ the *normal space* to F at a. Then $T_aM = T_aF \oplus iV$ for the corresponding tube manifold M, and N_aF can be canonically identified with the normal space $N_aM = E/T_aM$ of M in E. Define the map $\ell_a : T_aF \times T_aF \to N_aF$ in the following way: For $v, w \in T_aF$ choose a smooth map $f : V \to V$ with $f(a) = w$ and $f(x) \in T_xF$ for all $x \in F$ (actually it suffices to choose such an f only in a small neighborhood of a). Then put

$$\ell_a(v, w) := f'(a)(v) \mod T_aF\,, \tag{3.1}$$

where the linear operator $f'(a) \in \mathrm{End}(V)$ is the derivative of f at a. One shows that ℓ_a does not depend on the choice of f and is a symmetric bilinear map. We mention that if V is provided with the flat Riemannian metric and N_aF is identified with $T_aF^{\perp}$ then ℓ is nothing but the second fundamental form of F (see the subsection II.3.3 in [15]). The form ℓ_a can also be read off from local equations for F, more precisely, suppose that $U \subset V$ is an open subset, W is a real vector space and $h : U \to W$ is a real-analytic submersion with $F = h^{-1}(0)$. Then the derivative $h'(a) : V \to W$ induces a linear isomorphism $N_aF \cong W$ and modulo this identification ℓ_a is nothing but the second derivative $h''(a) : V \times V \to W$ restricted to $T_aF \times T_aF$.

By

$$K_aF := \{w \in T_aF : \ell_a(v, w) = 0 \ \text{ for all } \ v \in T_aF\}$$

we denote the *kernel* of ℓ_a. The manifold F is called *nondegenerate* at a if $K_aF = 0$ holds. The following statement follows directly from the definition of ℓ_a:

3.2 Lemma. *Suppose that* $\varphi \in \mathrm{End}(V)$ *satisfies* $\varphi(x) \in T_xF$ *for all* $x \in F$. *Then* $\varphi(a) \in K_aF$ *if and only if* $\varphi\big(T_aF\big) \subset T_aF$.

Lemma 3.2 applies in particular for $\varphi = \mathrm{id}$ in case F is a *cone,* that is, $rF = F$ for all real $r > 0$. More generally, we call the submanifold $F \subset V$ *conical* if $x \in T_xF$ for all $x \in F$. Then $\mathbb{R}a \subset K_aF$ holds for all $a \in F$.

In the remaining part of this section we explain how the CR-structure of the tube manifold M is related to the real objects ℓ_a, TF, KF, K^kF. In general it is quite hard to check whether a given CR-manifold M is k-nondegenerate at a point $a \in M$. For affinely homogeneous tube manifolds, however, there are simple criteria, see Propositions 3.4 and 3.6. We start with some preparations.

For every $a \in F \subset M$

$$H_aM = T_aF \oplus iT_aF \ \subset \ E$$

is the holomorphic tangent space at a, and E/H_aM can be canonically identified with $N_aF \oplus iN_aF$. It is easily seen that the Levi form $\mathcal{L}_a$ of M at a, compare (2.1), is nothing but the sesquilinear extension of the form ℓ_a from $T_aF \times T_aF$ to $H_aM \times H_aM$. In particular,

$$K_aM = K_aF \oplus iK_aF$$

is the *Levi kernel* of M at a. In case the dimension of K_aF does not depend on $a \in F$ these spaces form a subbundle $KF \subset TF$. In that case to every $v \in K_aF$ there exists a smooth function $f : V \to V$ with $f(a) = v$ and $f(x) \in K_xF$ for all $x \in F$, i.e., the tangent vector v extends to a smooth section in KF. In any case, let us define inductively linear subspaces K_a^kF of T_aF as follows:

3.3 Definition. For every real-analytic submanifold $F \subset V$ and every $k \in \mathbb{N}$ put

(i) $K_a^0F := T_aF$,

(ii) $K_a^{k+1}F$ is the space of all vectors $v \in K_a^kF$ such that there is a smooth mapping $f : V \to V$ with $f'(a)(T_aF) \subset K_a^kF$, $f(a) = v$ and $f(x) \in K_x^kF$ for all $x \in F$.

It is clear that $K_a^1F = K_aF$ holds. Let us call F *uniformly degenerate* if for every $k \in \mathbb{N}$ the dimension of K_a^kF does not depend on $a \in F$. In that case for every $v \in K_a^kF$ the outcome of the condition $f'(a)(T_aF) \subset K_a^kF$ in (ii) does not depend on the choice of the smooth mapping $f : V \to V$ satisfying $f(a) = v$ and $f(x) \in K_x^kF$ for all $x \in F$. For instance, F is of uniform degeneracy if F is locally affinely homogeneous, that is, if there exists a Lie algebra $\mathfrak{a}$ of affine vector fields on V such that every $\xi \in \mathfrak{a}$ is tangent to F and such that the canonical evaluation map $\mathfrak{a} \to T_aF$ is surjective for every $a \in F$. Clearly, if F is locally affinely homogeneous in the above sense then the corresponding tube manifold $M = F \oplus iV$ is locally homogeneous as CR-manifold.

We identify every smooth map $f : V \to V$ with the corresponding smooth vector field $\xi = f(x)\partial/\partial x$ on V. Our computations in the following are considerably simplified by the obvious fact that every smooth vector field ξ on V has a unique smooth extension to E that is invariant under all translations $z \mapsto z + iv$, $v \in V$. In case ξ is tangent to $F \subset V$ the extension satisfies $\xi_z \in H_zM$ for all $z \in M$.

In case $F \subset V$ is uniformly degenerate in a neighbourhood of $a \in F$ we call F *k-nondegenerate* at a if $K_a^kF = 0$ and k is minimal with respect to this property. As a consequence of [14], compare the last lines therein, we state:

3.4 Proposition. *Suppose that F is uniformly degenerate in a neighbourhood of $a \in F$. Then the corresponding tube manifold $M = F \oplus iV$ is k-nondegenerate as CR-manifold at $a \in M$ if and only if F is k-nondegenerate at a in the above defined sense.*

3.5 Corollary. *Suppose that F is uniformly degenerate, $\dim(F) \geq 2$ and $K_xF = \mathbb{R}x$ holds for every $x \in F$. Then F is 2-nondegenerate at every point.*

Proof. The condition $K_xF = \mathbb{R}x$ implies $0 \notin F$, otherwise the uniform degeneracy of F would be violated. The map $f = \mathrm{id}$ has the property $f(x) \in K_xF$ for every $x \in F$. Hence, the relation $f'(x)(T_xF) = T_xF \not\subset K_xF$ implies $x \notin K_x^2F$ and thus $K_x^2F = 0$. □

For locally affinely homogeneous submanifolds $F \subset V$ the spaces K_a^kF can easily be computed.

3.6 Proposition. *Suppose that $\mathfrak{a}$ is a linear space of affine vector fields on V such that every $\xi \in \mathfrak{a}$ is tangent to F and the canonical evaluation mapping $\mathfrak{a} \to T_aF$ is a linear isomorphism. Then, given any $k \in \mathbb{N}$, the vector $v \in K_a^kF$ is in $K_a^{k+1}F$ if and only if for every $\xi = h(x)\partial/\partial x \in \mathfrak{a}$ the relation $\lambda^h(v) \in K_a^kF$ holds, where $\lambda^h := h - h(0)$ is the linear part of ξ.*

Proof. By the implicit function theorem, there exist open neighbourhoods Y of $0 \in \mathfrak{g}$ and X of $a \in M$ such that $g(y) := \exp(y)a$ defines a diffeomorphism $g : Y \to X$. Define the smooth map $f : X \to V$ by $f(g(y)) = \mu_y(v)$, where μ_y is the linear part of the affine transformation $\exp(y)$. Then $f(a) = v$ and $f(x) \in K_x^kF$ for every $x \in X$. The claim now follows from $f'(a)\big(g'(0)\xi\big) = \lambda^h(v)$ for every $\xi \in \mathfrak{g}$ and λ^h the linear part of h. □

It is easily seen that a necessary condition for M being minimal as CR-manifold is that F is not contained in an affine hyperplane of V. A sufficient condition is that the image of the form ℓ_a, see (3.1), spans the full normal space N_aF at every $a \in F$.

For later use in Proposition 5.8 we state

3.7 Lemma. *Suppose that $F \subset V$ is a submanifold such that for every $c \in V$ with $c \neq 0$ there exists a linear transformation $\lambda \in \mathrm{GL}(V)$ with $\lambda(F) = F$ and $\lambda(c) \neq c$ (this condition is automatically satisfied if F is a cone). Then for $M = F \oplus iV$ the CR-automorphism group $\mathrm{Aut}(M)$ has trivial center.*

Proof. Let an element in the center of $\mathrm{Aut}(M)$ be given and let $h : U \to E$ be its holomorphic extension to an appropriate connected open neighbourhood U of M. Since h commutes with every translation $z \mapsto z + iv$, $v \in V$, it is a translation itself: Indeed, for $a \in F$ fixed and $c := h(a) - a$ the translation $\tau(z) := z + c$ coincides with h on $a + iV$ and hence on U by the identity principle. In case $c \neq 0$ choose $\lambda \in \mathrm{GL}(V)$ as in the above assumption. Then h commutes with $\lambda \in \mathrm{Aut}(M)$ and $\lambda(c) = c$ gives a contradiction, showing $h(z) \equiv z$. □

For fixed complex vector space E as above let us denote by $\mathfrak{P}$ the complex Lie algebra of all polynomial holomorphic vector fields $f(z)\partial/\partial z$ on E, that is, $f : E \to E$ is a polynomial map. Then $\mathfrak{P}$ has the $\mathbb{Z}$-grading

$$\mathfrak{P} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{P}_k\,, \qquad [\mathfrak{P}_k, \mathfrak{P}_l] \subset \mathfrak{P}_{k+l}\,, \tag{3.8}$$

where $\mathfrak{P}_k$ is the k-eigenspace of $\mathrm{ad}(\delta)$ for the Euler field $\delta := z\partial/\partial z \in \mathfrak{P}$. Clearly, $\mathfrak{P}_k$ is the subspace of all $(k+1)$-homogeneous vector fields in $\mathfrak{P}$ if $k \geq 1$ and is 0 otherwise.

Now assume that $F \subset F$ is a real analytic submanifold, $M = F \oplus iV$ is the corresponding tube manifold and $a \in F$ is a given point. For $\mathfrak{g} := \mathfrak{hol}(M, a)$ then put

$$\mathfrak{g}_k := \mathfrak{g} \cap \mathfrak{P}_k \quad \text{for all} \quad k \in \mathbb{Z}\,.$$

Then $\{iv\partial/\partial z : v \in V\} \subset \mathfrak{g}_{-1}$, and equality holds if M is holomorphically nondegenerate. Every $g \in \mathrm{Aut}(M, a)$ induces a Lie algebra automorphism $\Theta := g_*$ of $\mathfrak{g}$ in a canonical way: In terms of local holomorphic representatives in suitable open neighbourhoods of a in E

$$\Theta(f(z)\partial/\partial z) = g'(g^{-1}z)(f(g^{-1}z)) \tag{3.9}$$

holds. This implies immediately

3.10 Lemma. *$g \mapsto g_*$ defines a group monomorphism $\mathrm{Aut}(M, a) \hookrightarrow \mathrm{Aut}(\mathfrak{g})$.*

Proof. Suppose that $\Theta := g_* = \mathrm{id}$ for a $g \in \mathrm{Aut}(M, a)$. Then Θ extends to a complex Lie automorphism of $\mathfrak{l} := \mathfrak{g} + i\,\mathfrak{g} \subset \mathfrak{hol}(E, a)$ and leaves $\mathfrak{P}_{-1} \subset \mathfrak{l}$ element wise fixed. This implies that g is represented by a translation of E. But $g(a) = a$ then gives $g = \mathrm{id}$. □

4. Tube manifolds over cones

In this section we always assume that the submanifold $F \subset V$ is conical (that is, $x \in T_x F$ for every $x \in F$) and that $a \in F$ is a given point. Then, for $M := F \oplus iV$, the Lie algebra $\mathfrak{g} := \mathfrak{hol}(M, a)$ contains the Euler vector field $\delta := z\partial/\partial z$.

4.1 Lemma. *Suppose that $\mathfrak{g}$ has finite dimension. Then $\mathfrak{g} \subset \mathfrak{P}$ and for $\mathfrak{g}_k := \mathfrak{g} \cap \mathfrak{P}_k$*

$$\mathfrak{g} = \bigoplus_{k \geq -1} \mathfrak{g}_k\,, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l} \quad \text{and} \quad \mathfrak{g}_{-1} = \{iv\partial/\partial z : v \in V\}\,. \tag{4.2}$$

In particular, every $f(z)\partial/\partial z \in \mathfrak{g}$ is a polynomial vector field on $E = V \oplus iV$ with $f(iV) \subset iV$.

Proof. Consider $\mathfrak{l} := \mathfrak{g} \oplus i\,\mathfrak{g} \subset \mathfrak{hol}(E, a)$, which contains the vector field $\eta := (z-a)\partial/\partial z$. We first show $\mathfrak{l} \subset \mathfrak{P}$: Fix an arbitrary $\xi := f(z)\partial/\partial z \in \mathfrak{l}$. Then in a certain neighbourhood of $a \in E$ there exists a unique expansion $\xi = \sum_{k\in\mathbb{N}} \xi_k$, where $\xi_k = p_k(z-a)\partial/\partial z$ for a k-homogeneous polynomial map $p_k : E \to E$. It is easily verified that the vector field ad$(\eta)\xi \in \mathfrak{l}$ has the expansion ad$(\eta)\xi = \sum_{k\in\mathbb{N}}(k-1)\xi_k$. Now assume that for $d := \dim(\mathfrak{l})$ there exist indices $k_0 < k_1 < \ldots < k_d$ such that $\xi_{k_l} \neq 0$ for $0 \leq l \leq d$. Since the Vandermonde matrix $\big((k_l-1)^j\big)$ in non-singular, we get that the vector fields $(\mathrm{ad}\,\eta)^j\xi = \sum_{k\in\mathbb{N}}(k-1)^j\xi_k$, $0 \leq j \leq d$, are linearly independent in $\mathfrak{l}$, a contradiction. This implies $\xi \in \mathfrak{P}$ as claimed.
Since $\mathfrak{g} \subset \mathfrak{P}$ has finite dimension, every $\xi \in \mathfrak{g}$ is a finite sum $\xi = \sum_{k=-1}^{m} \xi_k$ with $\xi_k \in \mathfrak{P}_k$ and $m \in \mathbb{N}$ not depending on ξ. For every polynomial $p \in \mathbb{R}[X]$ then $p(\mathrm{ad}\,\delta)\xi 1 = \sum_{k=-1}^{m} p(k)\xi_k$ shows $\xi_k \in \mathfrak{g}_k$ for all k, that is, $\mathfrak{g} = \oplus\,\mathfrak{g}_k$. The identity $\mathfrak{g}_{-1} = \{iv\partial/\partial z : v \in V\}$ follows from the fact that $\mathfrak{g}_{-1}$ is totally real in $\mathfrak{P}_{-1}$and this implies $f(iV) \subset iV$ for all $f(z)\partial/\partial z \in \mathfrak{g}_k$ by $[\mathfrak{g}_{-1}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}$ and induction on k. □

4.3 Proposition. *Assume that $\mathfrak{g} := \mathfrak{hol}(M, a)$ has finite dimension and that $F' \subset V$ is a further conical submanifold with tube manifold $M' = F' \oplus iV$. Then for every $a' \in F'$ every CR-isomorphism $(M, a) \to (M', a')$ of manifold germs is rational.*

Proof. Let $\mathfrak{l} := \mathfrak{g} + i\,\mathfrak{g}$ and $\mathfrak{l}' := \mathfrak{g}' + i\,\mathfrak{g}'$ for $\mathfrak{g}' := \mathfrak{hol}(M', a')$. Fix a CR-isomorphism $(M, a) \to (M', a')$. This is represented by a biholomorphic mapping $U \to U'$ with $g(a) = a'$ and $g(U \cap M) = U' \cap M'$ for suitable connected open neighbourhoods U, U' of $a, a' \in E$. Then g induces a Lie algebra isomorphism $\Theta = g_* : \mathfrak{l} \to \mathfrak{l}'$, whose inverse is given by

$$\Theta^{-1}\big(f(z)\partial/\partial z\big) = g'(z)^{-1} f\big(g(z)\big)\,\partial/\partial z\,. \tag{4.4}$$

Lemma 4.2 shows that $\mathfrak{l}$ and $\mathfrak{l}'$ consist of polynomial vector fields. Consequently there exist *polynomial* maps $p : E \to E$ and $q : E \to \mathrm{End}(E)$ such that

$$\Theta^{-1}\big(z\partial/\partial z\big) = p(z)\partial/\partial z \quad \text{and} \quad \Theta^{-1}\big(e\partial/\partial z\big) = \big(q(z)e\big)\,\partial/\partial z$$

for all $e \in E$. Then (4.4) implies $g'(z)^{-1} = q(z)$ and $g'(z)^{-1}g(z) = p(z)$, that is,

$$g(z) = q(z)^{-1}p(z)$$

in a neighbourhood of $a \in E$. □

4.5 Remark. The tube $\mathcal{M}$ over the future light cone shows that in Proposition 4.6 ‘rational’ cannot always be replaced by ‘affine’.

Recall that $\mathfrak{aut}(M, a) \subset \mathfrak{g} = \mathfrak{hol}(M, a)$ is defined as the isotropy subalgebra at a and $\mathrm{Aut}(M, a)$ is the CR-automorphism group of the manifold germ (M, a), the so called *stability group.*

4.6 Proposition. *The following conditions are equivalent in case* $\mathfrak{g} = \mathfrak{hol}(M,a)$ *has finite dimension.*

(i) $\mathfrak{g}_1 = 0$.

(ii) $\mathfrak{g}_k = 0$ *for all* $k \geq 1$.

(iii) *Every germ in* $\mathrm{Aut}(M,a)$ *can be represented by a linear transformation* $g \in \mathrm{GL}(V) \subset \mathrm{GL}(E)$ *with* $g(a) = a$.

(iv) *The tangential representation* $h \mapsto h'(a)$ *induces a group monomorphism* $\mathrm{Aut}(M,a) \hookrightarrow \mathrm{GL}(V)$.

Each of these conditions is satisfied if $\mathfrak{aut}(M,a) = 0$. *On the other hand, if (iv) is satisfied then the image of the group monomorphism is contained in the subgroup*

$$\{g \in \mathrm{GL}(V) : g\,\mathfrak{g}_0 = \mathfrak{g}_0\, g,\ g(a) = a \ \text{ and } \ g(T_aF) = T_aF\},$$

where $\mathfrak{g}_0$ *is considered in the natural way as linear subspace of* $\mathrm{End}(V)$.

Proof. Let $\mathfrak{l} := \mathfrak{g} + i\,\mathfrak{g} \subset \mathfrak{hol}(E,a)$ and $\mathfrak{l}_k := \mathfrak{g}_k + i\,\mathfrak{g}_k$ for all k.
(i) $\Longrightarrow$ (ii) A direct check shows that, given any $\xi \in \mathfrak{l}$, the equality $[\mathfrak{l}_{-1}, \xi] = 0$ implies $\xi \in \mathfrak{l}_{-1}$. Suppose $\mathfrak{l}_k \neq 0$ for some $k > 1$ and that k is minimal with respect to this property. Consequently, $[\mathfrak{l}_{-1}, \mathfrak{l}_k] \subset \mathfrak{l}_{k-1} = 0$ would imply $\mathfrak{l}_k \subset \mathfrak{l}_{-1}$, i.e., since $\mathfrak{l}_{-1} \cap \mathfrak{l}_k = 0$, we would have $\mathfrak{l}_k = 0$. This contradicts our assumption $\mathfrak{l}_k \neq 0$.
(ii) $\Longrightarrow$ (iii) Then $\eta := (z-a)\partial/\partial z \in \mathfrak{l}$ and $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{k}$, where $\mathfrak{k} := \{\xi \in \mathfrak{l} : \xi_a = 0\}$ is the kernel of $\mathrm{ad}(\eta)$ in $\mathfrak{l}$. Every germ in $\mathrm{Aut}(M,a)$ can be represented by a locally biholomorphic map $h : U \to E$ with $h(a) = a$, where U is an open neighbourhood of a in E. The Lie algebra automorphism $\Theta \in \mathrm{Aut}(\mathfrak{l})$ induced by h leaves the isotropy subalgebra $\mathfrak{k} \subset \mathfrak{l}$ invariant. But η is the unique vector field $\xi \in \mathfrak{k}$ such that $\mathrm{ad}(\xi)$ induces the negative identity on the factor space $\mathfrak{l}/\mathfrak{k}$. Therefore $\Theta(\eta) = \eta$ and hence also $\Theta(\mathfrak{l}_{-1}) = \mathfrak{l}_{-1}$ since $\mathfrak{l}_{-1}$ is the (-1)-eigenspace of $\mathrm{ad}(\eta)$. From $\Theta(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$ we derive $h(z) = g(z) + c$ with $g \in \mathrm{GL}(V)$ and $c = a - g(a) \in V$. Taking the commutator of $t\cdot\mathrm{id}$ with h we see that the translation $z \mapsto z + (t-1)c$ leaves M invariant for all $t > 0$. Differentiation by t gives $c\partial/\partial z \in \mathfrak{g}_{-1}$, that is, $c = 0$ and $h = g$.
(iii) $\Longrightarrow$ (iv) This is trivial.
(iv) $\Longrightarrow$ (i) Let $\xi \in \mathfrak{g}_1$ be an arbitrary vector field. Then there exists a unique symmetric bilinear map $b : E \times E \to E$ with $\xi = b(z,z)\partial/\partial z$. Now $(\mathrm{ad}\, ia\partial/\partial z)^2\xi = -2b(a,a)\partial/\partial z \in \mathfrak{g}$, that is, $\eta := h(z)\partial/\partial z$ is in $\mathfrak{aut}(M,a)$, where $h(z) := b(z,z) - b(a,a)$. For every $t \in \mathbb{R}$ therefore the transformation $\psi_t := \exp(t\eta) \in \mathrm{Aut}(M,a)$ has derivative $\psi_t'(a) = \exp(th'(a)) \in \mathrm{GL}(E)$ in a. But $\psi_t'(a) \in \mathrm{GL}(V)$ by (iv) and thus $2b(a,v) = h'(a)v \in V$ for all $v \in V$. On the other hand $b(a,v) \in iV$ by Lemma 3.7, implying $\psi_t'(a) = \mathrm{id}$ for all $t \in \mathbb{R}$. By the injectivity of the tangential representation therefore η_t does not depend on t and we get $\xi = 0$. This proves the equivalence of (i) – (iv).

Now suppose $\mathfrak{aut}(M,a) = 0$ and that there exists a non-zero vector field $\xi \in \mathfrak{g}_1$. Then $\xi_a \in iV$ and there exists $\eta \in \mathfrak{g}_{-1}$ with $\xi - \eta \in \mathfrak{aut}(M,a)$, a contradiction. Finally, assume (iv) and let $g \in \mathrm{Aut}(M,a)$ be arbitrary. By (iii) then $g \in \mathrm{GL}(V)$ and $g(a) = a$. The induced Lie algebra automorphism Θ maps every $\lambda \in \mathfrak{g}_0$ to $g\lambda g^{-1} \in \mathfrak{g}_0$, compare (4.4). □

We close the section with an example that illustrates how in certain cases $\mathfrak{g} = \mathfrak{hol}(M,a)$ can be explicitly computed with the above results. We note that we do not know a single example with $\dim\mathfrak{g} < \infty$ and $\dim\mathfrak{g}_k > \dim\mathfrak{g}_{-k}$ for some $k \in \mathbb{N}$.

4.7 Example. Fix integers $p \geq q \geq 0$ with $n := p + q \geq 3$ and a real number α with $\alpha^2 \neq \alpha$. Then for $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$

$$F = F^\alpha_{p,q} := \Big\{x \in \mathbb{R}^n_+ : \sum_{j=1}^n \varepsilon_j x_j^\alpha = 0\Big\}, \quad \varepsilon_j := \begin{cases} 1 & j \leq p \\ -1 & \text{otherwise} \end{cases},$$

is a hypersurface in $V := \mathbb{R}^n$. Furthermore, F is a cone and therfore $\dim K_aF \geq 1$ for every $a \in F$. On the other hand, the second derivative at a of the defining equation for F gives a non-degenerate symmetric

bilinear form on $V \times V$, whose restriction to $T_aF \times T_aF$ then has a kernel of dimension ≤ 1. Therefore $\dim(K_aF) = 1$ for every $a \in F$ and by Corollary 3.5 the CR-manifold $M = M^{\alpha}_{p,q} := F^{\alpha}_{p,q} \oplus iV$ is everywhere 2-nondegenerate (compare Example 4.2.5 in [8] for the special case $n = \alpha = 3$). Since M as hypersurface is also minimal, $\mathfrak{g} = \mathfrak{hol}(M, a)$ has finite dimension.

For the special case $\alpha = 2$ and $q = 1$ the above cone $F = F^2_{n-1,1}$ is an open piece of the future light cone

$$\{x \in \mathbb{R}^n : x_1^2 + \ldots + x_{n-1}^2 = x_n^2,\ x_n > 0\}$$

in n-dimensional space-time, which is affinely homogeneous. In [14] it has been shown that for the corresponding tube manifold M the Lie algebra $\mathfrak{g} = \mathfrak{hol}(M, a)$ is isomorphic to $\mathfrak{so}(n, 2)$ for every $a \in M$. In case $q > 1$ the following result is new:

Case $\boldsymbol{\alpha} = 2$: Consider on $\mathbb{C}^n$ the symmetric bilinear form $\langle z|w\rangle := \sum \varepsilon_j z_j w_j$. Then F is an open piece of the hypersurface $\{x \in \mathbb{R}^n : x \neq 0,\ \langle x|x\rangle = 0\}$, on which the reductive group $\mathbb{R}^* \cdot \mathrm{O}(p, q)$ acts transitively. Therefore, $\mathfrak{s}_0 := \mathbb{R}\delta \oplus \mathfrak{so}(p, q)$ is contained in $\mathfrak{g}_0$. One checks that

$$\mathfrak{s} := \mathfrak{g}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1, \qquad \mathfrak{s}_1 := \left\{\big(2i\langle c|z\rangle z - i\langle z|z\rangle c\big)\,\partial/\partial z : c \in \mathbb{R}^n\right\}$$

is a Lie subalgebra of $\mathfrak{g}$. The radical $\mathfrak{r}$ of $\mathfrak{s}$ is ad(δ)-invariant and hence of the form $\mathfrak{r} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$ for $\mathfrak{r}_k := \mathfrak{r} \cap \mathfrak{g}_k$. From $\mathfrak{so}(p, q)$ semisimple we conclude $\mathfrak{r}_0 \subset \mathbb{R}\delta$. But δ cannot be in $\mathfrak{r}$ since otherwise $\mathfrak{g}_{-1} \subset \mathfrak{r}$ would give the false statement $[\mathfrak{g}_{-1}, \mathfrak{s}_1] \subset \mathbb{R}\delta$. Therefore $\mathfrak{r}_0 = 0$, and $[\mathfrak{g}_{-1}, \mathfrak{r}_1] = [\mathfrak{r}_{-1}, \mathfrak{s}_1] = 0$ implies $\mathfrak{r} = 0$. Now Proposition 3.8 in [14] implies $\mathfrak{g} = \mathfrak{s}$, and, in particular, that $\mathfrak{g}$ has dimension $\binom{n+2}{2}$. In fact, it can be seen that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p + 1, q + 1)$.

Case $\boldsymbol{\alpha}$ an integer $\geq$ 3: Then F is an open piece of the real-analytic submanifold

$$S := \Big\{x \in \mathbb{R}^n : x \neq 0 \quad \text{and} \quad \sum_{j=1}^{n} \varepsilon_j x_j^{\alpha} = 0\Big\} \tag{4.8}$$

which is connected in case $q > 1$ and has two connected components otherwise. For every $x \in \mathbb{R}^n$ let $d(x) \in \mathbb{N}$ be the cardinality of the set $\{j : x_j = 0\}$. It is easily seen that $\dim K_xS = 1 + d(x)$ holds for every $x \in S$. Now consider the group

$$\mathrm{GL}(F) := \{g \in \mathrm{GL}(V) : g(F) = F\}\,.$$

Every $g \in \mathrm{GL}(F)$ leaves S and hence also $H := \{x \in \mathbb{R}^n : d(x) > 0\}$ invariant, that is, g is the product of a diagonal with a permutation matrix. Inspecting the action of $\mathrm{GL}(F)$ on $\{c \in \overline{F} : d(c) = n - 2\}$ we see that $\mathrm{GL}(F)$ as group is generated by $\mathbb{R}_+ \cdot \mathrm{id}$ and certain coordinate permutations. As a consequence, $\mathfrak{g}_0 = \mathbb{R}\delta$ and, by a simplified version of the argument following (5.7), $\mathfrak{g}_k = 0$ for all $k > 0$ is derived. In particular, $\dim \mathfrak{g} = n + 1 < \dim M$ for the tube manifold $M = F \oplus iV$, that is, M is not locally homogeneous. For $n = 3$ this gives an alternative proof for Proposition 6.36 in [9].

4.9 Proposition. *Let $a, a' \in F = F^{\alpha}_{p,q}$ be arbitrary points. Then in case $3 \leq \alpha \in \mathbb{N}$ the CR-manifold germs (M, a) and (M, a') are CR-equivalent if and only if $a' \in \mathrm{GL}(F)a$.*

Proof. Suppose that $g : (M, a) \to (M, a')$ is an isomorphism of CR-manifold germs. From $\mathfrak{g}' = \mathfrak{g}'_{-1} \oplus \mathbb{R}\delta$ for $\mathfrak{g}' := \mathfrak{hol}(M', a')$ we conclude that g is represented by a linear transformation in $\mathrm{GL}(V)$ that we also denote by g. But then $g(F) \subset S$ with S defined in (4.8). Because $g(F)$ has empty intersection with H we actually have $g(F) \subset F$. Replacing g by its inverse we get the opposite inclusion, that is $g \in \mathrm{GL}(F)$. $\square$

5. Levi degenerate CR-manifolds associated with an endomorphism

Obviously, 2 is the lowest CR-dimension for which there exist homogeneous CR-manifolds that are Levi degenerate but not holomorphically degenerate. In the following we present a large class of such manifolds that contains all linearly homogeneous conical tube manifolds of CR-dimension 2.

5.1 Construction. Throughout this section let V be a real vector space of dimension $n \geq 3$ and $d \geq 1$ an integer with $d \leq n-2$. Let furthermore $\varphi \in \mathrm{End}(V)$ be a fixed endomorphism and $\mathfrak{h} \subset \mathrm{End}(V)$ the linear span of all powers φ^k for $0 \leq k \leq d$. Then $H := \exp(\mathfrak{h}) \subset \mathrm{GL}(V)$ is an abelian subgroup and for given $a \in V$ the orbit $F := H(a)$ is a cone and an immersed submanifold of V (not necessarily locally closed in case $n \geq 4$). The tube $M := F \oplus iV \subset E$ is an immersed submanifold of E and a homogeneous CR-manifold in a natural way.

5.2 Proposition. *Suppose that M is minimal. Then M is 2-nondegenerate and a is a cyclic vector of φ (that is, V is spanned by the vectors $\varphi^k(a)$, $k \geq 0$). Furthermore, M has CR-dimension $d{+}1$, and the Levi kernel K_aM has dimension 1.*

Proof. Let $W \subset V$ be the linear span of all vectors $\varphi^k(a)$, $k \geq 0$. Then $H \subset \mathbb{R}[\varphi]$ implies $H(a) \subset W$ and hence $W = V$ by the minimality of M. Therefore, a is a cyclic vector, and the $\varphi^k(a)$, $0 \leq k \leq d$, form a basis of the tangent space T_aF, that is, M has CR-dimension $d+1$. Lemma 3.2 gives $\mathbb{R}a \subset K_aF$ since F is a cone in V. For the proof of the opposite inclusion fix an arbitrary $w \in T_aF$ with $w \notin \mathbb{R}a$. Then $w = \sum_0^m c_j\varphi^j(a)$ with $c_m \neq 0$ for some $m \geq 1$, and $\varphi^{d-m+1}(w) \notin T_aF$ shows $w \notin K_aF$ by Proposition 3.6. Therefore, M is 2-nondegenerate by Corollary 3.5. □

From linear algebra it is clear that φ has a cyclic vector if and only if minimal and characteristic polynomial of φ coincide. Furthermore, if $a, b \in V$ both are cyclic vectors of φ then there exists a transformation $g \in \mathbb{R}[\varphi] \subset \mathrm{End}(V)$ with $b = g(a)$. But g commutes with every element of the group $H = \exp(\mathfrak{h})$ and hence maps the orbit $H(a)$ onto $H(b)$. Since $H(b)$ is not contained in a hyperplane of V, necessarily g is in $\mathrm{GL}(V)$, i.e. all cyclic vectors of φ give locally CR-equivalent tube manifolds.

For M in Proposition 5.2 the Lie algebra $\mathfrak{g}_0$ contains the $(d{+}1)$-dimensional Lie algebra $\mathfrak{h}$. We claim, that actually for φ in 'general position' $\mathfrak{g}_0 = \mathfrak{h}$ holds. Indeed, suppose that φ is diagonalizable with eigenvalues $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. For $N := \{1, 2, \ldots, n\}$ and every subset $I \subset N$ of cardinality $\#I = d{+}1$ put

$$\Delta_I := \left\{(\alpha_k - \alpha_j, \alpha_k^2 - \alpha_j^2, \ldots, \alpha_k^d - \alpha_j^d) \in \mathbb{C}^d : k \in N,\, j \in I\right\},$$

which obviously contains the origin $0 \in \mathbb{C}^d$.

5.3 Proposition. *Suppose that M is minimal, φ is diagonalizable and that*

$$\bigcap_{\#I=d+1} \Delta_I = \{0\}\,. \tag{5.4}$$

Then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{h}$ and $\mathfrak{aut}(M, a) = 0$.

Proof. By the minimality of M the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ are mutually distinct. The Lie algebra $\mathfrak{g}_0$ consists of all operators $\mu \in \mathrm{End}(V)$ that are tangent to $F = H(a)$, or equivalently, such that for every $t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d$ there exist real coefficients $r_0, r_1, \ldots, r_d$ with

$$\mu(e^{t_1\varphi + t_2\varphi^2 + \ldots + t_d\varphi^d}a) = (r_0 + r_1\varphi + \ldots + r_d\varphi^d)e^{t_1\varphi + t_2\varphi^2 + \ldots + t_d\varphi^d}a\,. \tag{5.5}$$

Since the orbit $H(a)$ contains a basis of V the matrix μ is uniquely determined by the function tuple $r_0, \ldots, r_d$ on $\mathbb{R}^d$. Going to the complexification E of V we may assume that $E = \mathbb{C}^n$ and that φ is given

by the diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$. Since $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ is a cyclic vector of φ we have $a_j \neq 0$ for every j. Then (5.5) for $\mu = (\mu_{jk}) \in \mathbb{C}^{n\times n}$ reads

$$(5.6) \qquad r_0 + \alpha_j r_1 + \alpha_j^2 r_2 + \ldots + \alpha_j^d r_d = a_j^{-1} \sum_{k=1}^{n} \mu_{jk} a_k e^{(\alpha_k-\alpha_j)t_1+(\alpha_k^2-\alpha_j^2)t_2+\ldots+(\alpha_k^d-\alpha_j^d)t_d}$$

for all $j \in N$. For every subset $I \subset N$ with $\#I = d{+}1$ the system of all linear equations (5.6) with $j \in I$ has a unique solution for $r_0, \ldots, r_d$ since the coefficient matrix is of Vandermonde type. In particular, every r_p is a complex linear combination of functions $e^{\beta_1 t_1+\beta_2 t_2+\ldots+\beta_d t_d}$ with $(\beta_1, \ldots, \beta_d) \in \Delta_I$. Note that given any set of pairwise different vectors $(\beta_1, \ldots, \beta_d)$ the corresponding functions $e^{\beta_1 t_1+\beta_2 t_2+\ldots+\beta_d t_d}$ are linearly independent. Consequently, if follows from (5.4) that actually every r_p is constant on $\mathbb{R}^d$. But this implies $\dim \mathfrak{g}_0 \le d+1 = \dim \mathfrak{h}$ and thus $\mathfrak{g}_0 = \mathfrak{h}$.
For the second claim let $\xi \in \mathfrak{g}_1$ be an arbitrary vector field. With respect to $E = \mathbb{C}^n$ and φ the diagonal matrix as before, we may write

$$(5.7) \qquad \xi = \sum_{j,k,p=1}^{n} c_p^{jk} z_j z_k \,\partial/\partial z_p\,, \qquad c_p^{jk} = c_p^{kj} \in \mathbb{C}\,.$$

Then

$$\xi_r := \left[\partial/\partial z_r, \xi\right] = 2\sum_{k,p} c_p^{kr} z_k \,\partial/\partial z_p \in \mathfrak{g}_0 \oplus i\,\mathfrak{g}_0 = \mathfrak{h} \oplus i\,\mathfrak{h}$$

for all $r \in N$ implies $c_p^{kr} = 0$ if $k \neq p$ and thus, by the symmetry in k and r, we have $\xi_r = 2c_r^{rr} z_r \,\partial/\partial z_r$. On the other hand, if $\lambda(z)\partial/\partial z \in \mathfrak{h} \oplus i\,\mathfrak{h}$ is non-zero then $\operatorname{rank}(\lambda) \ge n-d \ge 2$. But this implies $\xi_r = 0$ for all r, and in turn $\xi = 0$, i.e. $\mathfrak{g}_1 = 0$. By Proposition 4.6 therefore $\mathfrak{g}_k = 0$ holds for all $k > 0$. Finally, $\dim(\mathfrak{g}) = n+d+1 = \dim(M)$ implies $\mathfrak{aut}(M,a) = 0$. □

Notice that in case $n = \dim(V) = 3$ the condition (5.4) is equivalent to

$$\alpha_1 \neq (\alpha_2+\alpha_3)/2, \quad \alpha_2 \neq (\alpha_3+\alpha_1)/2 \quad \text{and} \quad \alpha_3 \neq (\alpha_1+\alpha_2)/2\,.$$

This will be applied in examples 6.2 and 6.4 (where 5.4 holds) and in example 6.1 (where 5.4 fails).

The conclusion in Proposition 5.3 also holds in many cases where φ is not diagonalizable, see e.g. Example 6.3 below. In any case, we get as a consequence of the above

5.8 Proposition. *Suppose that the tube M over the cone $F = H(a)$ is simply connected and that* Aut(M,a) *is the trivial group. Then the following properties hold:*

(i) *Let M' be an arbitrary homogeneous CR-manifold and $D \subset M$, $D' \subset M'$ non-empty domains. Then every real-analytic CR-isomorphism $h : D \to D'$ extends to a real-analytic CR-isomorphism $M \to M'$.*

(ii) *Let M' be an arbitrary locally homogeneous CR-manifold and $D' \subset M'$ a domain that is CR-equivalent to M. Then $D' = M'$.*

Proof. The group $G :=$ Aut(M) acts simply transitive on M and has trivial center by Lemma 3.7.
ad (i) We may assume $a \in D$. To every $g \in G$ with $g(a) \in D$ there exists a transformation $g' \in G' :=$ Aut(M') with $hg(a) = g'h(a)$. Because of Aut$(M', a') = \{\text{id}\}$ the transformation g' is uniquely determined by g and satisfies $hg = g'h$ in a neighbourhood of a. Since the Lie group G is simply connected $g \mapsto g'$ extends to a group homomorphism $G \to G'$ and h extends to a CR-covering map $h : M \to M'$. The deck transformation group $\Gamma := \{g \in G : gh = h\}$ is in the center of G, which is trivial by Lemma 3.7. Therefore, $h : M \to M'$ is a CR-isomorphism.
ad (ii) The proof is essentially the same as for Proposition 6.3 in [10]. □

The condition 'locally homogeneous' in Proposition 5.8.ii cannot be omitted. A counter example is given for every integer $\alpha \geq 3$ by the tube $M' \subset \mathbb{C}^3$ over

$$F' := \{x \in \mathbb{R}^3 : x_3 = x_1(x_2/x_1)^\alpha,\ x_1 > 0\}\,.$$

Then the tube M over $F := \{x \in F' : x_2 > 0\}$ is the Example 6.4 below that satisfies the assumption of Proposition 5.8.

The manifold M in Proposition 5.8 is simply connected, for instance, if there exist eigenvalues $\alpha_0, \alpha_1, \ldots, \alpha_d$ of φ such that $\det(A + \overline{A}) \neq 0$, where $A = (\alpha_j^k)_{0 \leq j,k \leq d}$ is the corresponding Vandermonde matrix. A criterion for the triviality of the stability group is given by the following statement.

5.9 Proposition. *Suppose that $a \in V$ is a cyclic vector of $\varphi \in \operatorname{End} V$ and that $\mathfrak{g}_1 = 0$ for $\mathfrak{g} = \mathfrak{hol}(M, a)$, $M = F \oplus iV$ and $F = H(a) \subset V$ as above. Then $\operatorname{Aut}(M, a) = \{\mathrm{id}\}$ and $\mathfrak{g}_0 = \mathfrak{h}$ if*

$$\{g\varphi g^{-1} : g \in \mathrm{GL}(V) \ \text{ with } \ g(a) = a\} \ \cap \ \mathfrak{g}_0 = \{\varphi\}\,.$$

Proof. Fix $g \in \operatorname{Aut}(M, a)$. By Proposition 4.6, $g \in \mathrm{GL}(V)$, $g(a) = a$, $g\varphi g^{-1} \in \mathfrak{g}_0$ and hence, by our assumption, $g\varphi = \varphi g$. But then $g\varphi^k(a) = \varphi^k g(a) = \varphi^k(a)$ for every k implies $g = \mathrm{id}$ since a is a cyclic vector. For the second claim note that by construction of M the restriction of the evaluation map $\varepsilon_a : \mathfrak{g} \to T_a M \ \cong \ \mathfrak{g}_{-1} \oplus \mathfrak{h}$ is surjective. Since $\operatorname{Aut}(M, a) = \{\mathrm{id}\}$ implies $\mathfrak{aut}(M, a) = 0 = \ker(\varepsilon_a)$, the claim follows. □

6. Homogeneous 2-nondegenerate CR-manifolds in dimension 5

In the following we present (up to local linear equivalence) all linearly homogeneous cones $F \subset \mathbb{R}^3$, for which $M = F \oplus i\mathbb{R}^3$ is 2-nondegenerate (that is, F is not contained in a hyperplane of $\mathbb{R}^3$). All these are obtained by the recipe of the preceding section: Choose a linear operator φ on $\mathbb{R}^3$ having a cyclic vector $a \in \mathbb{R}^3$. Then $\mathfrak{h} := \mathbb{R}\,\mathrm{id} + \mathbb{R}\varphi$ is an abelian Lie algebra and for the corresponding connected Lie subgroup $H \hookrightarrow \mathrm{GL}(3, \mathbb{R})$ the orbit $F := H(a)$ is a homogeneous cone with $M := F \oplus i\mathbb{R}^3$ Levi degenerate. In case φ' is another choice such that $\mathfrak{h}' = \mathbb{R}\,\mathrm{id} \oplus \mathbb{R}\varphi'$ is conjugate in $\mathrm{GL}(3, \mathbb{R})$ to $\mathfrak{h}$, then the spectra of φ, φ' differ in $\mathbb{C}$ only by an affine transformation $z \mapsto rz + s$ with $r, s \in \mathbb{R}$ and $r \neq 0$.

6.1 Example. Let $F = F_0 := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2,\ x_3 > 0\}$ be the future light cone. This occurs for the choice $\varphi := x_2 \partial/\partial x_1 - x_1 \partial/\partial x_2$ having spectrum $\{\pm i, 0\}$. The vector field $\varphi' := x_3 \partial/\partial x_2 + x_2 \partial/\partial x_3$ has spectrum $\{\pm 1, 0\}$ and generates with δ a 2-dimensional Lie algebra $\mathfrak{h}'$ that is tangent to F. An $\mathfrak{h}'$-orbit is the domain $F' := \{x \in F : x_1 > 0\}$ in F. The same happens with the nilpotent vector field

$$\varphi'' := \varphi + \varphi' = x_2 \partial/\partial x_1 + (x_3 - x_1) \partial/\partial x_2 + x_2 \partial/\partial x_3\,,$$

that generates with δ a 2-dimensional Lie algebra $\mathfrak{h}''$. Then $F'' := \{x \in F : x_2 < x_3\}$ is the unique open $\mathfrak{h}''$-orbit in F. For the tube $M = F \oplus i\mathbb{R}^3$ it is known that $\mathfrak{g} := \mathfrak{hol}(M, a)$ is a 10-dimensional simple Lie algebra, isomorphic to $\mathfrak{so}(2, 3)$, compare [14], [10]. The convex hull $\hat{F}$ of F is $\hat{F} = \{x \in \mathbb{R}^3 : x_3^2 \geq x_1^2 + x_2^2,\ x_3 > 0\}$.

6.2 Example. For $\alpha > 0$ let $F = F_\alpha \subset \mathbb{R}^3$ be the orbit of $(1, 0, 1)$ under the group of all linear transformations $x \mapsto e^s(\cos t\, x_1 - \sin t\, x_2, \sin t\, x_1 + \cos t\, x_2, e^{\alpha t} x_3)$, $s, t \in \mathbb{R}$. With $r := (x_1^2 + x_2^2)^{1/2}$, the manifold F is given in $\{x \in \mathbb{R}^3 : r > 0\}$ by the explicit equations

$$(*) \qquad x_3 = r \exp\left(\alpha \cos^{-1}(x_1/r)\right) = r \exp\left(\alpha \sin^{-1}(x_2/r)\right),$$

where locally always one of these suffices. A suitable choice is $\varphi = x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1 + \alpha x_3 \partial/\partial x_3$ with spectrum $\{\pm i, \alpha\}$. The convex hull of F is $\hat{F} = \{x \in \mathbb{R}^3 : x_3 > 0\}$. Notice that replacing α by $-\alpha$ would lead to a linearly equivalent cone.

6.3 Example. Let $F = F_\infty \subset \mathbb{R}^3$ be the orbit of $(1, 0, 1)$ under the group of all linear transformations $x \mapsto e^s(x_1, x_2 + tx_1, e^t x_3)$ with $s, t \in \mathbb{R}$, that is,

$$F = \left\{ x \in \mathbb{R}^3 : x_1 > 0,\ x_3 = x_1 e^{x_2/x_1} \right\}.$$

Here $\varphi = x_1 {}^{\partial}\!/\!{}_{\partial x_2} + x_3 {}^{\partial}\!/\!{}_{\partial x_3}$ has spectrum $0, 0, 1$ and is not diagonalizable.

6.4 Example. For $\alpha < -1$ let

$$F = F_\alpha := \{ x \in \mathbb{R}^3_+ : x_3 = x_1 (x_2/x_1)^\alpha \}.$$

Here $\varphi = x_2 {}^{\partial}\!/\!{}_{\partial x_2} + \alpha x_3 {}^{\partial}\!/\!{}_{\partial x_3}$ has distinct real eigenvalues $\{\alpha, 0, 1\}$. Notice that the limit case $\alpha = -1$ has already been discussed in 6.1.

For every F and $M = F \oplus i\mathbb{R}^3$ in the examples 6.2 – 6.4 consider for $\mathfrak{g} := \mathfrak{hol}(M, a)$ the decomposition (4.2). By a straightforward calculation it is seen that always $\mathfrak{g}_0 = \mathbb{R}\delta \oplus \mathbb{R}\varphi$ and $\mathfrak{g}_k = 0$ for $k > 0$ (for 6.2 and 6.4 this also follows from Proposition 5.3 and for 6.3 use Proposition 5.9). In particular, $\mathfrak{g}$ is a solvable Lie algebra of dimension 5 with commutator $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_{-1}$ of dimension 3. Including now also Example 6.1 let us for better distinction write $F = F_\alpha$, $M = M_\alpha$ and $a = a_\alpha$, where $\alpha \in \mathbb{R} \cup \{\infty\}$ satisfies $\alpha < -1$ or $\alpha \geq 0$.

6.5 Lemma. *The real Lie algebras* $\mathfrak{hol}(M_\alpha, a_\alpha)$ *are mutually non-isomorphic.*

Proof. Since $\mathcal{M} = M_0$ is the only manifold among all M_α such that $\mathfrak{hol}(M_\alpha, a_\alpha)$ is not of dimension 5 we restrict our attention to $M = M_\alpha$ and $a = a_\alpha$ with $\alpha \neq 0$. Then, as noted before, $\mathfrak{g}_{-1} = [\mathfrak{g}, \mathfrak{g}]$ for $\mathfrak{g} = \mathfrak{hol}(M, a)$, and $\mathfrak{g}_{-1} \oplus \mathbb{R}\delta$ is the set of all $\xi \in \mathfrak{g}$ such that the restriction $\rho(\xi) := \mathrm{ad}(\xi)|_{\mathfrak{g}_{-1}}$ acts as a multiple of the identity on $\mathfrak{g}_{-1}$. Therefore its complement $\mathfrak{g} \backslash (\mathfrak{g}_{-1} \oplus \mathbb{R}\delta)$ is an invariant of $\mathfrak{g}$.
Denote by Γ the group of all real affine transformations $z \mapsto rz + t$ with $r, t \in \mathbb{R}$ and $r \neq 0$. Then Γ also acts in a natural way on the space $\mathbb{C}^3/\mathfrak{S}_3$ of all unordered complex number triples. For every $\xi \in \mathfrak{g} \backslash (\mathfrak{g}_{-1} \oplus \mathbb{R}\delta)$ the operator $\rho(\xi)$ has precisely 3 (not necessarily distinct) complex eigenvalues and hence determines a point in $\mathbb{C}^3/\mathfrak{S}_3$. It is easy to see that the set Σ_α of all points in $\mathbb{C}^3/\mathfrak{S}_3$ obtained in this way by elements $\xi \notin (\mathfrak{g}_{-1} \oplus \mathbb{R}\delta)$ is a Γ-orbit and also is an invariant of the Lie algebra $\mathfrak{g} = \mathfrak{hol}(M_\alpha, a_\alpha)$. Since the map $\alpha \mapsto \Sigma_\alpha$ is injective, the claim follows. □

The surfaces $F = F_\alpha \subset \mathbb{R}^3$ in examples 6.1 – 6.4 are all cones. Cones belong to the class of (locally) ruled surfaces and are characterized by the property that all rules meet in a point. Another case of a ruled surface is if all rules are the tangents of a fixed curve in $\mathbb{R}^3$. By [7], p. 45, there is up to affine equivalence only one homogeneous ruled surface of this type that can be described as follows.

6.6 Example. Let S the union of all tangents to the *twisted cubic* $C := \{(s, s^2, s^3) \in \mathbb{R}^3 : s \in \mathbb{R}\}$, that is,

$$S = \{ x \in \mathbb{R}^3 : x_3^2 + 4x_2^3 - 6x_1x_2x_3 - 3x_1^2x_2^2 + 4x_1^3x_3 = 0 \}.$$

The two one-parameter groups

$$x \mapsto (e^t x_1, e^{2t} x_2, e^{3t} x_3), \qquad x \mapsto (x_1 + t, x_2 + 2tx_1 + t^2, x_3 + 3tx_2 + 3t^2 x_1 + t^3)$$

leave C and hence also S invariant. They generate a connected 2-dimensional group H of affine transformations that has three orbits in S. Besides C there are two further orbits in S that are interchanged by the involution $x \mapsto (-x_1, x_2, -x_3)$. Denote by F the H-orbit containing $a := (1, 0, 0)$. Then F is an affinely homogeneous surface in $\mathbb{R}^3$. Clearly, the Lie algebra $\mathfrak{h}$ of H is generated by the affine vector fields

$$\zeta = x_1 {}^{\partial}\!/\!{}_{\partial x_1} + 2x_2 {}^{\partial}\!/\!{}_{\partial x_2} + 3x_3 {}^{\partial}\!/\!{}_{\partial x_3}, \qquad \eta = {}^{\partial}\!/\!{}_{\partial x_1} + 2x_1 {}^{\partial}\!/\!{}_{\partial x_2} + 3x_2 {}^{\partial}\!/\!{}_{\partial x_3}.$$

Here, $T_aF = \{c \in \mathbb{R}^3 : c_3 = 0\}$ and $K_aF = \{c \in \mathbb{R}^3 : c_2 = c_3 = 0\}$ are easily checked with Lemma 3.2. With Proposition 3.6 we see that F is 2-nondegenerate. In particular, $\mathfrak{g} := \mathfrak{hol}(M,a)$ has finite dimension.

For every integer k denote by $\mathfrak{g}^{(k)}$ the k-eigenspace of $\mathrm{ad}(\zeta)$ in $\mathfrak{g}$. Then we have the $\mathbb{Z}$-gradation

$$\mathfrak{g} = \bigoplus_{k\ge -3} \mathfrak{g}^{(k)} \,. \tag{6.7}$$

It can be seen that $\mathfrak{g}^{(-3)} = \mathbb{R}i\partial/\partial z_3$, $\mathfrak{g}^{(-2)} = \mathbb{R}i\partial/\partial z_2$, $\mathfrak{g}^{(-1)} = \mathbb{R}i\partial/\partial z_1 \oplus \mathbb{R}\eta$, $\mathfrak{g}^{(0)} = \mathbb{R}\zeta$ and $\mathfrak{g}^{(k)} = 0$ for $k > 0$ (here ζ and η are in the unique way extended to complex affine vector fields on $\mathbb{C}^3$). This implies that $\mathfrak{hol}(M,a)$ is a solvable Lie algebra of dimension 5 with commutator ideal $[\mathfrak{g},\mathfrak{g}] = \bigoplus_{k<0}\mathfrak{g}^{(k)}$ of dimension 4. On the other hand, for $M = F\oplus i\mathbb{R}^3$ with $F\subset\mathbb{R}^3$ a cone from Example 6.1 – 6.4 the commutator of $\mathfrak{hol}(M,a)$ has either dimension 10 (Example 6.1) or dimension 3 (all the others). As a consequence of Lemma 6.5 we therefore get:

6.8 Proposition. *The manifolds $M = F\oplus i\mathbb{R}^3$, F a cone from examples* 6.1 – 6.6*, are all homogeneous* 2*-nondegenerate CR-manifolds. Furthermore, they are mutually locally CR-inequivalent.*

6.9 Lemma. *Let $M := F\oplus i\mathbb{R}^3$ for $F = H(a)\subset\mathbb{R}^3$ as in Example* 6.6*. Then* $\mathrm{Aut}(M,a)$ *is the trivial group.*

Proof. Let $\widetilde{M}$ be the image of M under the translation $z\mapsto z-a$, where $a = (1,0,0)$. It is enough to show that $\mathrm{Aut}(\widetilde{M},0)$ is the trivial group. Let $\mathfrak{l} := \widetilde{\mathfrak{g}} + i\widetilde{\mathfrak{g}}$ for $\widetilde{\mathfrak{g}} := \mathfrak{hol}(\widetilde{M},0)$ and put $\xi_j := \partial/\partial z_j$ for $j = 1,2,3$. From (6.7) and the explicit description of all $\mathfrak{g}^{(k)}$ we see that $\mathfrak{l} = \mathfrak{P}_{-1}\oplus\mathfrak{l}_0$ for $\mathfrak{l}_0 = \mathbb{C}\zeta\oplus\mathbb{C}\eta$ with

$$\zeta := z_1\partial/\partial z_1 + 2z_2\partial/\partial z_2 + 3z_3\partial/\partial z_3, \quad \eta := 2z_1\partial/\partial z_2 + 3z_2\partial/\partial z_3\,.$$

Now fix $g\in\mathrm{Aut}(\widetilde{M},0)$ and denote by $\Theta = g_*$ the induced Lie algebra automorphism of $\mathfrak{l}$. Then $\mathfrak{l}_0$ is Θ-invariant as isotropy subalgebra at the origin. Further Θ-invariant linear subspaces are $\mathfrak{n} := [\mathfrak{l},\mathfrak{l}]$ and thus also

$$\mathbb{C}\eta = \mathfrak{l}_0\cap\,\mathfrak{n},\ \ \mathbb{C}\xi_1 = K_0\widetilde{M},\ \ \mathbb{C}\xi_2 = H_0\widetilde{M}\cap[\mathfrak{n},\mathfrak{n}],\ \ \mathbb{C}\xi_3 = [\mathbb{C}\xi_2,\mathbb{C}\eta]\,.$$

As a consequence, there exist $s_1,s_2,s_3,u,v\in\mathbb{C}^*$ and $w\in\mathbb{C}$ such that $\Theta(\xi_j) = s_j\xi_j, \Theta(\eta) = u\eta$ and $\Theta(\zeta) = v\zeta + w\eta$. Then

$$s_1\xi_1 = \Theta(\xi_1) = \Theta([\xi_1,\zeta]) = [s_1\xi_1, v\zeta + w\eta] = s_1v\xi_1 + 2s_1w\xi_2$$

implies $v = 1$, $w = 0$ and hence $\Theta(\zeta) = \zeta$. Since $s_ji\xi_j$ is contained in $\widetilde{\mathfrak{g}}$ we get $s_j\in\mathbb{R}$ for $j = 1,2,3$. Also the vector fields $\zeta+\xi_1$ and $\eta+\xi_1+2\xi_2$ are contained in $\widetilde{\mathfrak{g}}$ and have Θ-image in $\widetilde{\mathfrak{g}}$, that is, $(1-s_1)\xi_1$ and $(1-u)\eta + 2(1-s_2)\xi_2$ are contained in $\widetilde{\mathfrak{g}}$. This implies $s_1 = s_2 = u = 1$. Finally, Θ applied to $[\xi_2,\eta] = 3\xi_3$ gives $s_3 = 1$. Therefore $\Theta = \mathrm{id}$ and Lemma 3.10 completes the proof. $\square$

For the tube $\mathcal{M}$ over the future light cone (that is Example 6.1) there exist infinitely many homogeneous CR-manifolds that are mutually globally CR-inequivalent but are all locally CR-equivalent to $\mathcal{M}$, compare [14]. In contrast to this, we have for the remaining examples:

6.10 Proposition. *Let $M = F\oplus iV$ with F from one of the examples* 6.2 – 6.6*. Then M is simply connected and* $\mathrm{Aut}(M)$ *is a solvable Lie group of dimension 5 acting transitively and freely on M. For every $a\in M$ the stability group* $\mathrm{Aut}(M,a)$ *is trivial and every homogeneous real-analytic CR-manifold M', that is locally CR-equivalent to M, is already globally CR-equivalent to M.*

Proof. It is easily checked that F and hence M is simply connected. In case F is a cone, the assumption in Proposition 5.9 is satisfied and the claim follows with Proposition 5.8. Therefore we may assume that F is the submanifold of Example 6.6. But then $\mathrm{Aut}(M,a)$ is the trivial group by Lemma 6.9 and $\mathrm{Aut}(M)$ has trivial center by Proposition 3.7. But then the proof of Proposition 5.8 also applies to this case. $\square$

With an argument from [10] together with 2.5.10 in [13] it can be seen that every continuous CR-function on $M = F \oplus iV$, $F \subset V$ a cone as above, has a unique continuous extension to the convex hull $\hat{M} = \hat{F} \oplus i\mathbb{R}^3$ of M which is holomorphic on the interior of $\hat{M}$ with respect to $\mathbb{C}^3$. In particular, if F belongs to Example 6.2, then every continuous CR-function on M is real-analytic.

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